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$D = 1$ Supergravity and Spinning Particles*

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Abstract

In this paper I review the multiplet calculus of $N = 1$, $D = 1$ local supersymmetry with applications to the construction of models for spinning particles in background fields, and models with space-time supersymmetry. New features include a non-linear realization of the local supersymmetry algebra and the coupling to anti-symmetric tensor fields of both odd and even rank. The non-linear realization allows the construction of a $D = 1$ cosmological-constant term, which provides a mass term in the equations of motion.

* Dedicated to Jurek Lukierski on the occasion of his 60th anniversary.

1 Worldline supersymmetry

Supersymmetry, as a symmetry between bosons and fermions, was discovered almost 25 years ago [1]-[3]. Apart from mathematical elegance, supersymmetry has the quality of improving the short-distance behaviour of quantum theories and has therefore been proposed as an ingredient of many models of physical phenomena, most often but not exclusively in the domain of particle physics and quantum gravity. Many of these applications are presently speculative, but it was realized already early that supersymmetric extensions of relativistic particle mechanics describe ordinary Dirac fermions [4]-[7]. Supersymmetric theories of this type are known as spinning particle models. They are useful providing low-energy descriptions for fermions in external fields [7]-[12] and path-integral expressions for perturbative amplitudes in quantum field theories [13]-[20]. They are also useful in studying aspects of higher-dimensional supersymmetric field theories and superstring models [21].

Like in string theory, in the discussion of supersymmetric point-particle models one has to distinguish between worldline supersymmetry, where the supersymmetry concerns transformations of the worldline parameters of the physical variables (proper time), and space-time supersymmetry which refers to supersymmetry in the target space of the physical variables. Both types of supersymmetry are encountered in the literature. In fact, in [22] a model was constructed possessing both types of supersymmetry simultaneously.

In the present paper I review the construction of pseudo-classical spinning particle models with worldline supersymmetry. To this end I present the multiplet calculus for local worldline supersymmetry ($N = 1$ supergravity in one dimension) and construct general lagrangians for $D = 1$ supersymmetric linear and non-linear σ -models with potentials; many elements of this formalism were developed in [24, 25]. The coupling to all kinds of background fields, including scalars, abelian and non-abelian vectors fields, gravity and anti-symmetric tensors is discussed in some detail. I finish with the construction of a model which exhibits space-time supersymmetry as well [22].

2 $D = 1$ supermultiplets

Supergravity models in $0 + 1$ space-time dimensions describe spinning particles. Indeed, the local supersymmetry and reparametrization invariance generate first-class constraints which after quantization can be identified with the Dirac and Klein-Gordon equations. Hence the quantum states of the model are spinorial wave functions for a fermion in a d -dimensional target space-time.

The models I construct below are general $N = 1$ supergravity actions in $D = 1$ with at most 2 proper-time derivatives, and the corresponding quantum theories. Higher- N models have been considered for example in [26, 27]. Extended target-

space supersymmetry has been studied in [28, 29].

In one dimension supersymmetry is realized off-shell¹ by a number of different sets of variables, the supermultiplets (superfields):

1. The gauge multiplet (e, χ) consisting of the einbein e and its superpartner χ , the gravitino; under infinitesimal local worldline super-reparametrizations, generated by the parameter-valued operator $\delta(\xi, \varepsilon)$ where $\xi(\tau)$ is the commuting parameter of translations and $\varepsilon(\tau)$ the anti-commuting parameter of supersymmetry, the multiplet transforms as

$$\delta e = \frac{d(\xi e)}{d\tau} - 2i\varepsilon\chi, \quad \delta\chi = \frac{d(\xi\chi)}{d\tau} + \frac{d\varepsilon}{d\tau}. \quad (1)$$

2. Scalar multiplets (x, ψ) , used to describe the position and spin co-ordinates of particles. The transformation rules are

$$\delta x = \xi \frac{dx}{d\tau} - i\varepsilon\psi, \quad \delta\psi = \xi \frac{d\psi}{d\tau} + \varepsilon \frac{1}{e} \mathcal{D}_\tau x, \quad (2)$$

where the supercovariant derivative is constructed with the gravitino as the connection:

$$\mathcal{D}_\tau x = \frac{dx}{d\tau} + i\chi\psi. \quad (3)$$

3. Fermionic multiplets (η, f) with Grassmann-odd η and even f . The f -component is most often used as an auxiliary variable, without dynamics of itself. The transformation properties under local super-reparametrizations are

$$\delta\eta = \xi \frac{d\eta}{d\tau} + \varepsilon f, \quad \delta f = \xi \frac{df}{d\tau} - i\varepsilon \frac{1}{e} \mathcal{D}_\tau \eta. \quad (4)$$

The supercovariant derivative is formed using the same recipe as before:

$$\mathcal{D}_\tau \eta = \frac{d\eta}{d\tau} - \chi f. \quad (5)$$

¹The term *off shell* implies that the supersymmetry algebra is realized without using dynamical constraints like the equations of motion.

4. A non-linear multiplet consisting of a single fermionic component σ with the transformation rules

$$\delta\sigma = \xi \frac{d\sigma}{d\tau} + \varepsilon - i\varepsilon \frac{1}{e} \sigma \mathcal{D}_\tau \sigma, \quad (6)$$

with the supercovariant derivative

$$\mathcal{D}_\tau \sigma = \frac{d\sigma}{d\tau} - \chi + \frac{i}{e} \chi \sigma \frac{d\sigma}{d\tau}. \quad (7)$$

On any component of any multiplet the commutator of two infinitesimal variations with parameters $(\xi_{1,2}, \varepsilon_{1,2})$ results in an infinitesimal transformation with parameters (ξ_3, ε_3) given by

$$\begin{aligned} [\delta(\xi_2, \varepsilon_2), \delta(\xi_1, \varepsilon_1)] &= \delta(\xi_3, \varepsilon_3), \\ \xi_3 &= \xi_1 \frac{d\xi_2}{d\tau} - \xi_2 \frac{d\xi_1}{d\tau} - \frac{2i}{e} \varepsilon_1 \varepsilon_2, \\ \varepsilon_3 &= \xi_1 \frac{d\varepsilon_2}{d\tau} - \xi_2 \frac{d\varepsilon_1}{d\tau} + \frac{2i}{e} \varepsilon_1 \varepsilon_2 \chi. \end{aligned} \quad (8)$$

For the non-linear representation σ the proof requires use of the supersymmetry variation

$$\delta(\varepsilon) \left(\frac{1}{e} \mathcal{D}_\tau \sigma \right) = -i\varepsilon \frac{1}{e} \mathcal{D}_\tau \left(\frac{1}{e} \sigma \mathcal{D}_\tau \sigma \right) = -i\varepsilon \frac{1}{e} \sigma \mathcal{D}_\tau \left(\frac{1}{e} \mathcal{D}_\tau \sigma \right). \quad (9)$$

Then a simple result is obtained:

$$\delta(\varepsilon) \left(\frac{1}{e} \sigma \mathcal{D}_\tau \sigma \right) = \varepsilon \frac{1}{e} \mathcal{D}_\tau \sigma. \quad (10)$$

It follows that each of these multiplets is a representation of the same local supersymmetry algebra, and this algebra closes off-shell. However, the parameters of the resulting transformation depend on the components of the gauge multiplet (e, χ) , indicating that the algebra of infinitesimal transformations is a *soft* commutator algebra, rather than an ordinary super Lie-algebra.

Among the representations discussed, the gauge multiplet and the non-linear multiplet have manifestly non-linear transformation rules. The variations of the other two multiplets are linear in the components of these multiplets. For this reason the scalar and fermionic multiplets are called *linear* representations of local supersymmetry, although some of the coefficients depend on the gauge variables (e, χ) .

3 Multiplet calculus

The linear representations (scalar and fermionic) satisfy some simple addition and multiplication rules; this tensor calculus has been developed in [25]. The rules can also be formulated in terms of $D = 1$ superfields [24]. As concerns addition, any linear multiplets of the same type can be added component by component with arbitrary real or complex coefficients. The linearity of the transformation rules then guarantees the sum to be a multiplet of the same type.

The multiplication rules are also simple. There are 3 different product formula's:

1. The product of two scalar multiplets $\Sigma = (x, \psi)$, $\Sigma' = (x', \psi')$ is a scalar multiplet

$$\Sigma \times \Sigma' = \Sigma'' = (xx', x\psi' + x'\psi). \quad (11)$$

This rule can be extended to arbitrary powers of scalar multiplets, for example:

$$\Sigma^n = (x^n, nx^{n-1}\psi). \quad (12)$$

In this way one can define functions of scalar multiplets by power series expansions.

2. The product of a scalar multiplet $\Sigma = (x, \psi)$ and a fermionic multiplet $\Phi = (\eta, f)$ is a fermionic multiplet

$$\Sigma \times \Phi = \Phi' = (x\eta, xf - i\psi\eta). \quad (13)$$

3. The product of two fermionic multiplets is a scalar multiplet:

$$-i\Phi \times \Phi' = \Sigma' = (-i\eta\eta', f\eta' - f'\eta). \quad (14)$$

Next I introduce the operation of derivation of scalar and fermionic multiplets; the super-derivative on linear multiplets is a Grassmann-odd linear operator turning a scalar multiplet into a fermionic multiplet, and vice-versa, with the following components:

$$\begin{aligned} \mathcal{D}\Sigma &= \Phi' = \left(\psi, \frac{1}{e}\mathcal{D}_\tau x\right); \\ \mathcal{D}\Phi &= \Sigma' = \left(f, \frac{1}{e}\mathcal{D}_\tau \eta\right). \end{aligned} \quad (15)$$

On product multiplets this super-derivative satisfies the Leibniz rule, with in particular the result

$$\mathcal{D}\Sigma^n = n\mathcal{D}\Sigma \times \Sigma^{n-1}. \quad (16)$$

The super-derivative satisfies an operator algebra similar to the supersymmetry algebra:

$$\mathcal{D}^2 = \frac{1}{e} \mathcal{D}_\tau, \quad (17)$$

where \mathcal{D}_τ is the supercovariant proper-time derivative on components, encountered before in eqs.(3) and (5).

4 Invariant actions

1. Invariant actions can be constructed for the linear as well as the non-linear multiplets. As there exists no intrinsic curvature in $D = 1$, there is no invariant action for the gauge multiplet involving the einbein, but there is a very simple action for the gravitino, namely

$$S_\Lambda = \int d\tau i\Lambda\chi, \quad (18)$$

where Λ is a constant. The equation of motion for this action by itself is not consistent (it requires Λ to vanish); but this is changed if one adds other terms to the action, like the ones discussed below. Also, one can replace χ in the action by $d\chi/d\tau$, but then the action becomes a total derivative.

A cosmological-constant like action can be constructed with the help of the non-linear multiplet; it reads

$$S_{nl}(\sigma) = \int d\tau (e - i\chi\sigma - i\sigma\mathcal{D}_\tau\sigma). \quad (19)$$

This is also the kinetic action for the fermionic σ variable, which in view of the anti-commuting nature of σ can be only linear in proper-time derivatives. Note, that the non-linear nature of the multiplet allows one to rescale the variable σ and thereby change the relative co-efficients between the various terms in the action. A rescaling of σ by a factor $1/c$ gives the action

$$S_{nl}(\sigma; c) = \int d\tau \left(e - \frac{2i}{c} \chi\sigma - \frac{i}{c^2} \sigma\dot{\sigma} \right), \quad (20)$$

where I have introduced the dot notation for ordinary proper-time derivatives. Of course, the rescaling also changes the non-linear transformation rule for σ under supersymmetry to

$$\delta_c\sigma = \xi \frac{d\sigma}{d\tau} + c\varepsilon - \frac{i\varepsilon}{ce} \sigma\mathcal{D}_\tau\sigma. \quad (21)$$

Combining the actions S_Λ and S_{nl} in such a way as to get standard normalization of the fermion kinetic term for σ leads to the action (with the dimension of \hbar)

$$\begin{aligned} S_{grav} &= mcS_\Lambda - \frac{mc^2}{2} S_{nl} \\ &= m \int d\tau \left(ic\Lambda\chi - \frac{c^2}{2} e + ic\chi\sigma + \frac{i}{2} \sigma\dot{\sigma} \right), \end{aligned} \quad (22)$$

where m is a parameter with the dimension of mass and c has the dimension of a velocity. The Euler-Lagrange equations for the fermions χ and σ then give

$$\sigma = \Lambda, \quad \chi = \frac{1}{c} \frac{d\sigma}{d\tau} = 0. \quad (23)$$

Thus the constant Λ is like a vacuum expectation value of the fermionic variable σ . However, the variation (21) of σ is such that for non-zero c it can be gauged away completely by a supersymmetry transformation. Therefore it does not represent a true physical degree of freedom. Of course, this seems to contradict the equation of motion (23), but we observe that also the equation for the einbein is inconsistent, requiring the constant c to vanish. Again, these problems are solved by adding further terms to the action. In applications c usually represents the velocity of light, which can conveniently be taken as unity ($c = 1$).

2. Next we turn to a formula for the construction of invariant actions for linear multiplets. Given a fermionic multiplet $\Phi = (\eta, f)$, an action invariant under local supersymmetry transformations is

$$S_{lin} = \int d\tau (ef - i\chi\eta). \quad (24)$$

Note, that in eqs.(19) and (24) the integrand itself is not invariant, but transforms into a total proper-time derivative:

$$\delta S_{lin} = \int d\tau \frac{d}{d\tau} (-i\varepsilon\eta), \quad (25)$$

and the same for δS_{nl} with $\eta \rightarrow \sigma$. Eq.(25) shows, that δS vanishes for variations which are zero on the endpoints.

Eq.(24) can be applied to the construction of actions for scalar multiplets if one applies an odd number of super-derivatives so as to obtain a composite fermionic multiplet, sometimes called the (fermionic) prepotential. A simple example of this construction is the free kinetic action for a scalar multiplet, constructed from the composite fermionic multiplet

$$\Phi_{kin} = \frac{1}{2} \mathcal{D}^2 \Sigma \times \mathcal{D} \Sigma, \quad (26)$$

Inserting the components of this multiplet into the action formula (24) gives

$$S_{kin} = \int d\tau \left(\frac{1}{2e} \dot{x}^2 + \frac{i}{2} \psi \dot{\psi} + \frac{i}{e} \chi \psi \dot{x} \right). \quad (27)$$

If one extends this formula to d free multiplets $\Sigma^\mu = (x^\mu, \psi^\mu)$, $\mu = 1, \dots, d$, then using the appropriate minkowski metric it becomes the action for a free spinning particle in d -dimensional space-time.

This is a special case of the most general action involving only scalar multiplets and quadratic in proper-time derivatives of the bosonic co-ordinates \dot{x}^μ : the $D = 1$ (non-)linear σ -model in a d -dimensional target space, constructed from a fermionic multiplet

$$\Phi[g] = \frac{1}{2} g_{\mu\nu}(\Sigma) \times \mathcal{D}^2 \Sigma^\mu \times \mathcal{D} \Sigma^\nu. \quad (28)$$

Here $g_{\mu\nu}(\Sigma)$ is a symmetric tensor in the space of the scalar multiplets, which can be interpreted as a metric on the target manifold. Using $\Phi[g]$ in the action formula (24) gives the component expression

$$S_{kin}[g] = \int d\tau \left(\frac{1}{2e} g_{\mu\nu}(x) \dot{x}^\mu \dot{x}^\nu + \frac{i}{2} g_{\mu\nu}(x) \psi^\mu D\psi^\nu + \frac{i}{e} g_{\mu\nu}(x) \chi \psi^\mu \dot{x}^\nu \right), \quad (29)$$

where D denotes a target-space covariant derivative

$$D\psi^\mu = \dot{\psi}^\mu + \dot{x}^\lambda \Gamma_{\lambda\nu}^{\mu} \psi^\nu, \quad (30)$$

with $\Gamma_{\kappa\nu}^{\mu}(x)$ the Riemann-Christoffel connection. The action $S_{kin}[g]$ is manifestly covariant in the target space. The symmetries of this action have been investigated in detail in [30, 31], with applications to special target manifolds like Schwarzschild space-time and Taub-NUT in [31, 32, 12].

The simplest action for scalar multiplets involves only one super-derivative. It starts from the general fermionic multiplet (super 1-form)

$$\Phi[A] = A_\mu(\Sigma) \times \mathcal{D} \Sigma^\mu, \quad (31)$$

with $A_\mu(\Sigma)$ an abelian vector field on the target space. Inserting the components of $\Phi[A]$ in the action formula (24) gives the result

$$S_{vec}[A] = \int d\tau \left(A_\mu(x) \dot{x}^\mu - \frac{ie}{2} F_{\mu\nu}(x) \psi^\mu \psi^\nu \right), \quad (32)$$

with $F_{\mu\nu}$ the field strength tensor of the abelian vector field $A_\mu(x)$.

A similar construction can be carried out using arbitrary odd super p -forms [24]. For example $p = 3$ gives

$$\Phi[H] = \frac{i}{3!} H_{\mu\nu\lambda}(\Sigma) \times \mathcal{D} \Sigma^\mu \times \mathcal{D} \Sigma^\nu \times \mathcal{D} \Sigma^\lambda. \quad (33)$$

This gives an action involving the anti-symmetric 3-tensor $H_{\mu\nu\lambda}(x)$:

$$S_{\text{odd}}[H] = \frac{i}{2}H_{\mu\nu\lambda}(x)\psi^\mu\psi^\nu\left(\dot{x}^\lambda + \frac{2i}{3}\chi\psi^\lambda\right) + \frac{e}{3!}\partial_\kappa H_{\mu\nu\lambda}(x)\psi^\kappa\psi^\mu\psi^\nu\psi^\lambda. \quad (34)$$

The inclusion of even p -forms is also possible, but requires one or more fermionic multiplets; details are given below.

In the same spirit one can find odd p -form extensions of the kinetic term

$$\Phi[G] = G_{\mu\nu_1\dots\nu_p}(\Sigma) \times \mathcal{D}^2\Sigma^\mu \times \mathcal{D}\Sigma^{\nu_1}\dots \times \Sigma^{\nu_p}. \quad (35)$$

For a discussion of these unconventional actions I refer to [24]. Finally, actions with higher powers of \mathcal{D}^2 and/or with \mathcal{D}^n ($n \geq 3$) lead to higher-derivative component lagrangians. I do not consider them here.

Combining the results for scalar fields, within the restrictions we have imposed the general action for scalar multiplets is of the form

$$S[\Sigma] = mS_{\text{kin}}[g] + qS_{\text{vec}}[A] + \alpha S_{\text{odd}}[H] + mcS_\Lambda - \frac{mc^2}{2}S_{nl}. \quad (36)$$

This action describes a spinning particle in background electro-magnetic and gravitational fields, with the possible inclusion of torsion for $\alpha \neq 0$. The first-class constraints obtained from the equation of motion for the einbein and gravitino are

$$\begin{aligned} m^2 g_{\mu\nu} (\dot{x}^\mu + i\chi\psi^\mu) (\dot{x}^\nu + i\chi\psi^\nu) &= -me^2 \left(mc^2 + iqF_{\mu\nu}(A)\psi^\mu\psi^\nu - \frac{\alpha}{3} F_{\kappa\mu\nu\lambda}(H)\psi^\kappa\psi^\mu\psi^\nu\psi^\lambda \right). \\ mg_{\mu\nu}\dot{x}^\mu\psi^\nu - \frac{i\alpha}{3}H_{\mu\nu\lambda}\psi^\mu\psi^\nu\psi^\lambda &= mce(\Lambda - \sigma). \end{aligned} \quad (37)$$

Here $F_{\mu\nu}(A)$ and $F_{\kappa\mu\nu\lambda}(H)$ are the field strenghts of the vector A_μ and 3-form $H_{\mu\nu\lambda}$, respectively. Eqs.(37) are the pseudo-classical equivalents of the Klein-Gordon and Dirac equations. Note that local supersymmetry can be used to chose a gauge $\sigma = \Lambda$ in which the expressions on both sides of the second equation vanish. If $c = 0$ (absence of S_{nl} and S_Λ) the particle is massless. With the inclusion of S_{nl} ($c \neq 0$) the particle acquires a non-zero mass.

If the kinetic terms are normalized in the standard way, the relative co-efficient q between the first two terms represents the electric charge of the spinning particle, as defined by the generalized Lorentz-force [11]. Then the anti-symmetric tensor $D^{\mu\nu} = q\psi^\mu\psi^\nu$ represents the electric and magnetic dipole moments. The terms involving the 3-form $H_{\mu\nu\lambda}(x)$ combine to form an anti-symmetric contribution to the Riemann-Christoffel connection, representing torsion indeed.

3. Finally we turn to the construction of actions involving elementary fermionic multiplets. To begin with, there is the simple action formula (24) linear in the

components of a single fermionic multiplet. It involves no proper-time derivatives, and therefore it can contribute only to potential terms. A natural and straightforward generalization of this action involving r fermionic multiplets Φ^i , $i = 1, \dots, r$, is constructed from the composite fermionic prepotential

$$\Phi[U] = U_i(\Sigma) \times \Phi^i, \quad (38)$$

with the $U_i(\Sigma)$ a set of scalar-multiplet valued potentials. The component action then is

$$S_{pot}[U] = \int d\tau \left(eU_i(x)f^i - iU_i(x)\chi\eta^i - ie\psi^\mu\partial_\mu U_i(x)\eta^i \right). \quad (39)$$

As the equation of motion for f^i requires all $U_i(x)$ to vanish, this action by itself is useful only to impose constraints on the target manifold. This conclusion is modified when additional (kinetic) terms are added to the action.

More complicated actions obtained using the multiplet calculus with both fermionic and scalar multiplets must have an odd total number of fermionic multiplets and super-derivatives. Therefore the next complicated type of action involves the product of two fermionic multiplets including a super-derivative. The general form of the fermionic prepotential is

$$\Phi[K] = \frac{1}{2} K_{ij}(\Sigma) \times \mathcal{D}\Phi^i \times \Phi^j, \quad (40)$$

with $K_{ij}(\Sigma)$ a scalar-multiplet valued symmetric matrix. The component action for this prepotential is

$$S_{ferm}[K] = \int d\tau \left(\frac{i}{2} K_{ij}(x)\eta^i\dot{\eta}^j + \frac{e}{2} K_{ij}(x)f^if^j - \frac{ie}{2}\psi^\mu\partial_\mu K_{ij}(x)f^i\eta^j \right). \quad (41)$$

It contains kinetic terms for the fermionic variables η^i , but the variables f^i only appear without derivatives and are auxiliary degrees of freedom. In combination with the potential term $S_{pot}[U]$ its elimination turns the constraints U_i into a true potential, allowing the bosonic variables to fluctuate around the solutions of the constraints $U_i(x) = 0$.

Other actions can be constructed by replacing some of the super-derivatives $\mathcal{D}\Sigma^\mu$ in the odd p -form prepotentials like $\Phi[H]$ (33) by fermionic multiplets. I give the details for the case $p = 3$. First consider a prepotential linear in fermionic multiplets:

$$\Phi[B] = -\frac{i}{2} B_{i\mu\nu}(\Sigma) \times \Phi^i \times \mathcal{D}\Sigma^\mu \times \mathcal{D}\Sigma^\nu. \quad (42)$$

The potentials $B_{i\mu\nu}(x)$ define r anti-symmetric tensors (2-forms) on the target space of the scalars. Thus this construction and its higher-rank generalizations

allows the inclusion of even p -forms in the action. Substitution in the linear-multiplet action S_{lin} , eq.(24), gives

$$S_{even}[B] = \int d\tau \left(-\frac{ie}{2} f^i B_{i\mu\nu}(x) \psi^\mu \psi^\nu - i\eta^i B_{i\mu\nu}(x) \psi^\mu \dot{x}^\nu + \frac{1}{2} \chi \eta^i B_{i\mu\nu}(x) \psi^\mu \psi^\nu \right. \\ \left. + \frac{e}{2} \eta^i \partial_\lambda B_{i\mu\nu}(x) \psi^\lambda \psi^\mu \psi^\nu \right). \quad (43)$$

Next consider the case of a quadratic expression in fermionic multiplets. The prepotential is

$$\Phi[V] = \frac{i}{2} V_{ij\mu}(\Sigma) \times \Phi^i \times \Phi^j \times \mathcal{D}\Sigma^\mu. \quad (44)$$

The vector field $V_{ij\mu}(x)$, anti-symmetric in $[ij]$, takes values in a gauge group $G \subseteq SO(r)$ for r even (in the quantum theory this is always the case [19]). Thus this action describes the coupling to Yang-Mills fields. After quantization the fermionic variables η^i generate a Clifford-algebra representation of the group G embedded in $SO(r)$ on the particle wave-functions. The explicit expression for the pseudo-classical action is

$$S_{YM}[V] = \int d\tau \left(\frac{i}{2} \eta^i \eta^j V_{ij\mu}(x) \dot{x}^\mu + \frac{e}{4} \eta^i \eta^j F_{ij\mu\nu}^{(0)}(x) \psi^\mu \psi^\nu + ie f^i V_{ij\mu}(x) \eta^j \psi^\mu \right). \quad (45)$$

Here $F_{\mu\nu}^{(0)}$ represents the abelian (linear) part of the field-strength for the vector field V_μ . The non-abelian part can be obtained by a proper choice of K_{ij} in S_{ferm} and subsequent elimination of the auxiliary variables f^i (see below).

Finally I consider the action cubic in fermionic multiplets:

$$\Phi[T] = \frac{i}{3!} T_{ijk}(\Sigma) \times \Phi^i \times \Phi^j \times \Phi^k. \quad (46)$$

The action constructed from this prepotential becomes

$$S_{int}[T] = \int d\tau \left(\frac{ie}{2} T_{ijk}(x) \eta^i \eta^j f^k + \frac{1}{3!} \chi T_{ijk}(x) \eta^i \eta^j \eta^k + \frac{e}{3!} \psi^\mu \partial_\mu T_{ijk}(x) \eta^i \eta^j \eta^k \right) \quad (47)$$

Comparison with the action $S_{pot}[U]$ shows, that this action represents the coupling to non-abelian scalar fields, where the generators of the group are again expressed in terms of the rank- r Grassmann algebra. Extensions of these results to higher-order forms are straightforward.

5 Applications

The actions constructed above can be used to describe spinning particles in a d -dimensional target space-time in various kinds of background fields: scalar fields, abelian and non-abelian vector fields, anti-symmetric tensor fields, rank-3 anti-symmetric torsion, etc. (Note that in four-dimensional space-time the rank-3 anti-symmetric tensor is dual to an axial vector field.) In this section I discuss some special examples which are particularly useful in physics applications.

1. Yukawa coupling. One of the simpler cases is that of a spinning particle in Minkowski space-time interacting with a scalar field. This situation is described by the kinetic action with $g_{\mu\nu} = \eta_{\mu\nu}$, the Minkowski metric, extended with the action S_{ferm} for the internal fermion variables in a flat background ($K_{ij} = \delta_{ij}$) and a potential term λS_{pot} , where λ is the coupling constant:

$$S_{Yuk} = mS_{kin}[\eta_{\mu\nu}] + S_{ferm}[\delta_{ij}] - \lambda S_{pot}[U]. \quad (48)$$

The full component action is

$$\begin{aligned} S_{Yuk} = \int d\tau \left(\frac{m}{2e} \dot{x}_\mu^2 + \frac{im}{2} \psi_\mu \dot{\psi}^\mu + \frac{im}{e} \chi \psi_\mu \dot{x}^\mu + \frac{i}{2} \eta_i \dot{\eta}^i + \frac{e}{2} f_i^2 \right. \\ \left. - \lambda e U_i(x) f^i + i\lambda U_i(x) \chi \eta^i + i\lambda e \psi^\mu \partial_\mu U_i(x) \eta^i \right) \end{aligned} \quad (49)$$

The auxiliary variables f^i can be eliminated using their algebraic Euler-Lagrange equation

$$f_i = \lambda U_i(x). \quad (50)$$

This gives the result

$$\begin{aligned} S_{Yuk} = \int d\tau \left(\frac{m}{2e} \dot{x}_\mu^2 + \frac{im}{2} \psi_\mu \dot{\psi}^\mu + \frac{im}{e} \chi \psi_\mu \dot{x}^\mu + \frac{i}{2} \eta_i \dot{\eta}^i - \frac{e}{2} \lambda^2 U_i^2 \right. \\ \left. + i\lambda U_i(x) \chi \eta^i + i\lambda e \psi^\mu \partial_\mu U_i(x) \eta^i \right). \end{aligned} \quad (51)$$

The constraints from varying the action with respect to the gauge variables are

$$\begin{aligned} m^2 (\dot{x}_\mu + i\chi \psi_\mu)^2 &= -me^2 (\lambda^2 U_i^2 - 2i\lambda \psi^\mu \partial_\mu U_i \eta^i), \\ m\dot{x}_\mu \psi^\mu + e\lambda U_i \eta^i &= 0. \end{aligned} \quad (52)$$

The model describes a spinning particle in a relativistic scalar potential $\lambda^2 U_i^2(x)$, which may be dynamical. If this field has a vacuum expectation value $\lambda^2 \langle U_i^2 \rangle =$

mc^2 , it generates a mass for the particle, showing it can act as a Higgs field. This mechanism of generating mass dynamically is an alternative to adding the non-linear multiplet action. However, in some sense the two mechanisms are the same, because the action for the linear fermionic multiplet (η, f) becomes identical with the non-linear multiplet action $S_{nl}(\sigma; c)$ if one imposes the constraint that $f = c$, a constant.

2. Yang-Mills coupling. A very interesting application from the point of view of particle physics is the case of a spinning particle (e.g., a quark or lepton) coupled to a vector gauge field $V_\mu(x)$ [7, 34, 35], a supersymmetric generalization of Wong's model [33]. Again, I consider ordinary Minkowski space-time and a flat internal space-time. Then adding the vector action:

$$S_{gauge} = mS_{kin}[\eta_{\mu\nu}] + S_{ferm}[\delta_{ij}] - gS_{YM}[V], \quad (53)$$

and eliminating the auxiliary variable f^i by its Euler-Lagrange equation

$$f_i = igV_{ij\mu}(x)\eta^j\psi^\mu, \quad (54)$$

the component action reads

$$S_{gauge} = \int d\tau \left(\frac{m}{2e} \dot{x}_\mu^2 + \frac{im}{2} \psi_\mu \dot{\psi}^\mu + \frac{im}{e} \chi \psi_\mu \dot{x}^\mu + \frac{i}{2} \eta_i \dot{\eta}^i + g \bar{V}_\mu \dot{x}^\mu - \frac{ige}{2} \bar{F}(V)_{\mu\nu} \psi^\mu \psi^\nu \right). \quad (55)$$

For convenience I have introduced here the Grassmann-algebra valued gauge field

$$\bar{V}_\mu = -\frac{i}{2} \eta^i \eta^j V_{ij\mu}, \quad (56)$$

and similarly for the field-strength:

$$\bar{F}(V)_{\mu\nu} = -\frac{i}{2} \eta^i \eta^j F_{ij\mu\nu}(V) = \partial_\mu \bar{V}_\nu - \partial_\nu \bar{V}_\mu - g [\bar{V}_\mu, \bar{V}_\nu]. \quad (57)$$

The equation of motion for a particle in a non-abelian background gauge field then becomes

$$m \frac{d^2 x^\mu}{d\tilde{\tau}^2} = g \bar{F}(V)^\mu{}_\nu \frac{dx^\nu}{d\tilde{\tau}} - \frac{ig}{2} D^\mu \bar{F}(V)_{\lambda\nu} \psi^\lambda \psi^\nu, \quad (58)$$

where $d\tilde{\tau} = ed\tau$ and D^μ is the gauge-covariant derivative. The first term represents the non-abelian Lorentz force, the second one the Stern-Gerlach term responsible for non-abelian spin-orbit interactions [11].

3. Gravity. The actions above can be easily generalized to include gravity, by using a general curved-space metric $g_{\mu\nu}(x)$ in the kinetic multiplet rather than the Minkowski metric $\eta_{\mu\nu}$. The internal-space metric $K_{ij}(x)$ however remains

flat. The only new feature is then to change the kinetic terms to the general form $S_{kin}[g]$, eq.(29).

4. *Anti-symmetric tensor coupling.* As a final example of the coupling of spinning particles to external fields we consider anti-symmetric rank-2 tensor fields $B_{i\mu\nu}(x)$ in curved space-time as well as internal space. The action to use is

$$S_{tensor} = mS_{kin}[g] + S_{ferm}[K] - yS_{even}[B], \quad (59)$$

where y is a coupling constant. The auxiliary variables f^i now satisfy the equation

$$K_{ij}f^j = \frac{i}{2} \left(\psi^\mu \partial_\mu K_{ij} \eta^j - y B_{i\mu\nu} \psi^\mu \psi^\nu \right). \quad (60)$$

To solve it, we assume that $K_{ij}(x)$ is invertible. Elimination of the auxiliary variables from the action then gives the component result

$$\begin{aligned} S_{tensor} = & \int d\tau \left(\frac{m}{2e} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \frac{im}{2} g_{\mu\nu} \psi^\mu D\psi^\nu + \frac{im}{e} g_{\mu\nu} \chi \psi^\mu \dot{x}^\nu + \frac{i}{2} K_{ij} \eta^i \dot{\eta}^j \right. \\ & - i\eta^i B_{i\mu\nu} \psi^\mu \dot{x}^\nu + \frac{1}{2} \chi \eta^i B_{i\mu\nu} \psi^\mu \psi^\nu + \frac{e}{2} \eta^i F_{i\lambda\mu\nu}(B) \psi^\lambda \psi^\mu \psi^\nu \\ & + \frac{e}{8} \eta^i \eta^j \left(\partial_\mu K \cdot K^{-1} \cdot \partial_\nu K \right)_{ij} \psi^\mu \psi^\nu - \frac{ye}{4} \eta^i \left(\partial_\lambda K \cdot K^{-1} \right)_i^j B_{j\mu\nu} \psi^\mu \psi^\nu \psi^\lambda \\ & \left. - \frac{y^2 e}{8} B_{\mu\nu} \cdot K^{-1} \cdot B_{\kappa\lambda} \psi^\mu \psi^\nu \psi^\kappa \psi^\lambda \right). \end{aligned} \quad (61)$$

Here $F_{i\mu\nu\lambda}(B) = 1/3 (\partial_\lambda B_{i\mu\nu} + \partial_\nu B_{i\lambda\mu} + \partial_\mu B_{i\nu\lambda})$ is the field-strength of the anti-symmetric tensor field. When the internal metric is flat: $K_{ij}(x) = \delta_{ij}$, considerable simplifications occur and the whole third line vanishes. In four dimensions the product $\psi^\mu \psi^\nu \psi^\kappa \psi^\lambda$ is proportional to $\varepsilon^{\mu\nu\kappa\lambda}$, and the last term is of the form $B \cdot K^{-1} \cdot \tilde{B}$, where the tilde denotes the dual tensor.

6 Space-time supersymmetry

In all previous examples the fermionic multiplets were used to represent internal degrees of freedom, connected with rigid or local internal symmetries. I conclude this paper with an application where the extra fermionic variables represent space-time degrees of freedom. This example is the spinning superparticle [22, 23], which possesses both (local) world-line and (rigid) target-space supersymmetry [36].

For simplicity, I consider only space-times which allow Majorana spinors ($d = 2, 3, 4 \bmod 8$). In such a space-time one can define, in addition to the usual co-ordinate multiplets Σ^μ , a spinor of real fermionic supermultiplets

$$\Psi_a = (\theta_a, h_a), \quad (62)$$

with $a = 1, \dots, 2^{\lfloor \frac{d}{2} \rfloor}$. More generally, in an arbitrary spinor basis we do not require reality, but the Majorana condition

$$\Psi = C\bar{\Psi}, \quad (63)$$

where $\bar{\Psi} = \Psi^\dagger \gamma_0$ is the Pauli conjugate spinor, and C is the charge conjugation matrix. Then the components (θ_a, h_a) define an anti-commuting and a commuting Majorana spinor in then target space-time, respectively. The super-derivative of this spinor of multiplets is defined as in eq.(15).

Introducing the Dirac matrices γ^μ in the d -dimensional target space-time, I next construct a d -vector of composite spinor multiplets

$$\Omega^\mu = \mathcal{D}\Sigma^\mu - \mathcal{D}\bar{\Psi}\gamma^\mu\Psi. \quad (64)$$

The components of these spinor multiplet are

$$\Omega^\mu \equiv (\omega^\mu, \Pi^\mu) = \left(\psi^\mu - \bar{h}\gamma^\mu\theta, \frac{1}{e}\mathcal{D}_\tau x^\mu + \frac{i}{e}\mathcal{D}_\tau\bar{\theta}\gamma^\mu\theta - \bar{h}\gamma^\mu h \right). \quad (65)$$

From the spinor supermultiplets Ω^μ it is straightforward to construct a fermionic prepotential which is a Lorentz scalar in target space-time:

$$\Phi[\Psi] = \frac{1}{2}\eta_{\mu\nu}\mathcal{D}\Omega^\mu \times \Omega^\nu. \quad (66)$$

The component action derived from this prepotential is

$$\begin{aligned} S_{super} &= \int d\tau \left[\frac{1}{2e} \left(\dot{x}_\mu - i\bar{\theta}\gamma_\mu\dot{\theta} - e\bar{h}\gamma_\mu h \right)^2 + \frac{i}{2} \left(\psi_\mu - \bar{h}\gamma_\mu\theta \right) \frac{d}{d\tau} \left(\psi_\mu - \bar{h}\gamma_\mu\theta \right) \right. \\ &\quad \left. + \frac{i}{e} \chi \left(\psi_\mu - \bar{h}\gamma_\mu\theta \right) \left(\dot{x}_\mu - i\bar{\theta}\gamma_\mu\dot{\theta} - e\bar{h}\gamma_\mu h \right) \right]. \\ &= \int d\tau \left[\frac{e}{2} \Pi_\mu^2 + \frac{i}{2} \omega_\mu \dot{\omega}^\mu \right]. \end{aligned} \quad (67)$$

A superfield derivation of this action has been presented in [37]. Similar models in two-dimensional space-time describing spinning strings were constructed in [38, 39, 40]. We observe that h is an auxiliary commuting Majorana spinor. Contrary to our previous actions, in S_{super} these auxiliary variables in general have a cubic and a quartic term, of the form $\gamma_\mu h \bar{h} \gamma^\mu h$ and $(\bar{h} \gamma_\mu h)^2$. However,

owing to the Fierz identities these terms vanish in four-dimensional space-time, where the auxiliary variables only appear quadratically.

The action S_{super} has a huge number of symmetries. Except for local worldline supersymmetry, I mention rigid target-space supersymmetry, under which the gauge multiplet (e, χ) is inert, whilst the linear multiplets Σ^μ and Ψ_a transform with an anti-commuting Majorana spinor parameter ϵ :

$$\begin{aligned}\delta x^\mu &= -i\bar{\theta}\gamma^\mu\epsilon, & \delta\psi^\mu &= \bar{h}\gamma^\mu\epsilon, \\ \delta\theta &= \epsilon, & \delta h &= 0.\end{aligned}\tag{68}$$

These transformations imply that the components of the multiplet $\Omega^\mu = (\omega^\mu, \Pi^\mu)$ in eq.(65) are invariant: $\delta\omega^\mu = \delta\Pi^\mu = 0$.

Then there is the Siegel invariance with anti-commuting spinor parameter κ on the worldline, which takes the form

$$\begin{aligned}\delta x^\mu &= i\bar{\theta}\gamma^\mu\gamma \cdot \Pi\kappa, & \delta\psi^\mu &= \frac{2i}{e}\bar{h}\gamma^\mu\theta\dot{\theta}\kappa + \bar{h}\gamma^\mu\gamma \cdot \Pi\kappa, \\ \delta\theta &= \gamma \cdot \Pi\kappa, & \delta h &= -\frac{2i}{e}h\dot{\theta}\kappa, \\ \delta e &= 4i\dot{\theta}\kappa, & \delta\chi &= 0.\end{aligned}\tag{69}$$

Under these variations the components (ω^μ, Π^μ) transform as

$$\delta\Pi^\mu = \frac{2i}{e}\dot{\theta}\gamma^\mu\gamma \cdot \Pi\kappa - \frac{4i}{e}\dot{\theta}\kappa\Pi^\mu, \quad \delta\omega^\mu = 0.\tag{70}$$

In addition there is a bosonic counterpart of the Siegel invariance [22] with commuting spinor parameter α :

$$\begin{aligned}\delta x^\mu &= 0, & \delta\psi^\mu &= \bar{\theta}\gamma^\mu\gamma \cdot \Pi\alpha - 2\bar{\theta}\gamma^\mu h\bar{h}\alpha, \\ \delta\theta &= 0, & \delta h &= \gamma \cdot \Pi\alpha - 2h\bar{h}\alpha, \\ \delta e &= -4e\bar{h}\alpha, & \delta\chi &= 0,\end{aligned}\tag{71}$$

resulting in

$$\delta\Pi^\mu = 2\bar{\alpha}\gamma^\mu\gamma \cdot \Pi h, \quad \delta\omega^\mu = 0.\tag{72}$$

Still other symmetries can be found for the massless spinning superparticle, which I do not discuss here. If one assigns the space-time supersymmetry transformation $\delta\sigma = 0$ to the non-linear fermion multiplet, addition of the mass term S_{grav} respects local world-line supersymmetry and space-time supersymmetry. However, in this case the Siegel transformations and their bosonic extension are no longer invariances of the model.

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References

- [1] Y.A. Gol’fand and E.P. Likhtman, JETP Lett. 13 (1971), 323
- [2] D.V. Volkov and V.P. Akulov, Phys. Lett. B46 (1973), 109
- [3] J. Wess and B. Zumino, Nucl. Phys. B70 (1974), 39
- [4] F.A. Berezin and M.S. Marinov, Ann. Phys. (NY) 104 (1977), 336
- [5] L. Brink, P. Di Vecchia and P. Howe, Nucl. Phys. B118 (1977), 76
- [6] R. Casalbuoni, Phys. Lett. B62 (1976), 49
- [7] A. Barducci, R. Casalbuoni and L. Lusanna, Nuov. Cim. 35A (1976), 377
Nucl. Phys. B124 (1977), 93; id. 521
- [8] A. Balachandran et al. Phys. rev. D15 (1977), 2308
- [9] M. Henneaux and C. Teitelboim, Ann. Phys. (NY) 143 (1982), 127
- [10] J.W. van Holten, Proc. Sem. Math. Structures in Field Theories 1986/87
CWI Syllabus vol. 26 (1990), 109
- [11] J.W. van Holten, Nucl. Phys. B356 (1991), 3; Physica A182 (1992), 279
- [12] J.W. van Holten, Phys. Lett. B342 (1995), 47
- [13] S.M. Shvartsman, Mod. Phys. Lett. A5 (1990), 943
- [14] M. Strassler, Nucl. Phys. B385 (1992), 145
- [15] D.G.C. McKeon, Ann. Phys. (NY) 224 (1993), 154
- [16] C. Schubert, Phys. Lett. B331 (1994), 69
- [17] M. Mondragon et al. Phys. Lett. B351 (1995), 200
- [18] E. d’Hoker and D. Gagné, hep-th/9508131
- [19] J.W. van Holten, preprint NIKHEF/95-050; hep-th/9508136; Nucl. Phys. B
(in press)

- [20] J. de Boer et al. SUNY preprint ITP-SB-95-32, hep-th/9509158
- [21] M. Green, J. Schwarz and E. Witten, *Superstring Theory*, 2 vols. (Cambridge Univ. Press, 1987)
- [22] S. Aoyama, J. Kowalski-Glikman, J. Lukierski and J.W. van Holten, Phys. Lett. B201 (1988) 487; Phys. Lett. B216 (1989), 133
- [23] J. Lukierski, Proc. Int. Conf. on *Selected topics in QFT and mathematical physics*, Liblice (Cz) (1989), 297
- [24] R.H. Rietdijk, Class. Quant. Grav. 9 (1992), 1395; Mod. Phys. Lett. A7 (1992), 881
- [25] J.M.A. de Vree, diploma thesis, Univ. of Amsterdam (1993)
- [26] E.A. Bergshoeff and J.W. van Holten, Phys. Lett. B226 (1989), 93
- [27] A.I. Pashnev and D.P. Sorokin, Kharkov preprint KFTI 90-31
- [28] J.A. de Azcarraga and J. Lukierski, Phys. Lett. B113 (1982) 170; Phys. Rev. D28 (1983), 1337
- [29] A. Frydryszak, Phys. Rev. D35 (1986), 2432
- [30] R.H. Rietdijk and J.W. van Holten, Class. Quant. Grav. 7 (1990), 247; J. Geom. Phys. 11 (1993), 559
- [31] G. Gibbons, R.H. Rietdijk and J.W. van Holten, Nucl. Phys. B404 (1993), 42
- [32] R.H. Rietdijk and J.W. van Holten, Class. Quant. Grav. 10 (1993), 575
- [33] S.K. Wong, Nuov. Cim. 65A (1970), 689
- [34] P.D. Jarvis and M.J. White, Phys. Rev. D43 (1991), 4121
- [35] N. Linden, A.J. Macfarlane and J.W. van Holten, preprint NIKHEF/95-049, DAMTP-95/37
- [36] M. Green and J. Schwarz, Phys. Lett. B136 (1984), 367
- [37] A.R. Kavalov and R.L. Mkrtychyan, preprint YERPHI-1068(31)-88
- [38] S.J. Gates, R. Brooks, I. Muhammed and S.J. Gates, Class. Quantum Grav. 3 (1986), 745
- [39] R. Brooks, Phys. Lett. B186 (1987), 313
- [40] J. Fisch, Phys. Lett. B219 (1989), 71